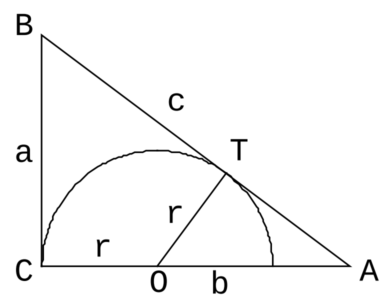
Q1. A right triangle has legs of lengths a and b and a hypotenuse of length c. A semicircle has its diameter on the side of length b and is tangent to the other two sides. Determine the radius of the semicircle in terms of a and b.



Solution: In triangle ABC, let T denote the point of tangency of side AB to the inscribed semicircle with center O and radius r. Since $\angle OTA = 90^\circ$, we see that $\sin(A) = r/(b-r)$; but we also have $\sin(A) = a/c$. Equating these two expressions for $\sin(A)$ yields $cr = ab - ar$. Solving for r gives $r = \dfrac{ab}{a+c} = \dfrac{ab}{a+\sqrt{a^2+b^2}}$.

Q2. In how many ways can twenty dollars be changed into dimes and quarters, with at least one of each coin used?

Solution: Let the number of dimes be denoted by D and the number of quarters by Q. By assumption, $1 \le D, Q$. Then $10D + 25Q = 2000$. Since the number of dimes must be a multiple of five to make exact change, let $D = 5P$ to get $10(5P) + 25Q = 2000$ or $2P + Q = 80$. Because $1 \le Q = 80 - 2P$, we must have $P \le 79/2$, and since P is a natural number this implies that $1 \le P \le 39$. Thus there are 39 ways total.

Q3. Three cards each have one of the digits from one to nine written on them. When the three cards are arranged in some order, they make a three digit number. The largest number that can be made plus the second largest number that can be made is 1233. What is the largest number that can be made?

Solution: Let the three digits be a, b and c. We may assume that $a \geq b \geq c$. Clearly all three digits cannot be the same, for then one would never get the three in the one's position after adding the two numbers.

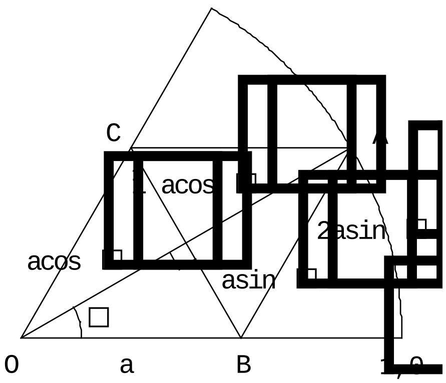
If all are distinct, so that $a > b > c$, then adding the numbers abc_{10} and acb_{10} and expanding in base ten gives the sum of $2a10^2 + (b+c)10 + (b+c)$ =1233. This implies that $b+c$ must equal 3 with no carries, which implies that $b = 2$ and $c = 1$. This gives $a = 6$ by inspection.

The case $a = b > c$, the same argument shows no a, c will work.

When $a > b = c$, the numbers being added are abb_{10} and bab_{10}. This cannot work either, for again one would never get the three in the one's position after adding the two numbers.

Thus 621 is the largest number, which when added to the second largest number 612, gives the sum of 1233.

Q4. Equilateral triangle ABC is inscribed in a sector of a circle, centered at the origin, of radius one so that there is one vertex on each radius and one vertex (say A) in the middle of the arc. The angle at the center is 2θ. What is the side length of triangle ABC?



Solution: Let a denote the length of OB. By Pythagorean's theorem we have $[2a\sin(\theta)]^2 = [a\sin(\theta)]^2 + [1 - a\cos(\theta)]^2$, which implies that $3a^2\sin^2(\theta) = [1 - a\cos(\theta)]^2$ or $\sqrt{3}a|\sin(\theta)| = 1 - a\cos(\theta)$. Since $0 < 2\theta < \pi$, that is, $0 < \theta < \pi/2$, we may remove the absolute value symbol. Solving for a we get

$$a = \frac{1}{\sqrt{3}\sin(\theta) + \cos(\theta)} = \frac{1}{2\sin(\theta + \pi/6)}.$$

Therefore the side length of the equilateral triangle is $2a\sin(\theta) = \dfrac{\sin(\theta)}{\sin(\theta + \pi/6)}$.

Q5. Determine the exact value of $f(1)+f(2)+f(3)+\cdots+f(2009)$, where

$$f(k)=\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}.$$

Solution: One forms a telescoping sum by noting that

$$f(k)=\left(\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)\left(\frac{k\sqrt{k+1}-(k+1)\sqrt{k}}{k\sqrt{k+1}-(k+1)\sqrt{k}}\right)$$

$$=\frac{k\sqrt{k+1}-(k+1)\sqrt{k}}{k^2(k+1)-(k+1)^2k}=\frac{(-1)\left[k\sqrt{k+1}-(k+1)\sqrt{k}\right]}{k(k+1)}$$

$$=\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}.$$

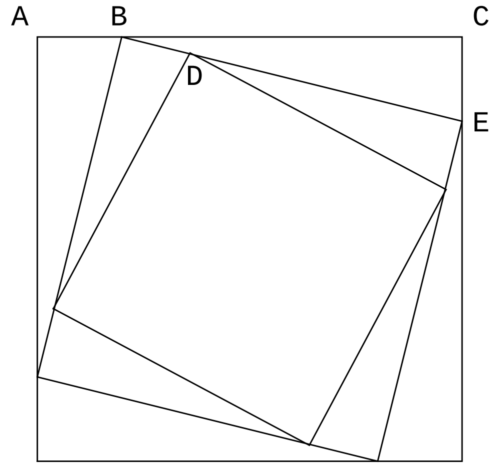
Thus

$$f(1)+f(2)+\cdots+f(2009)$$

$$=\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right)+\cdots+\left(\frac{1}{\sqrt{2008}}-\frac{1}{\sqrt{2009}}\right)+\left(\frac{1}{\sqrt{2009}}-\frac{1}{\sqrt{2010}}\right)$$

$$=1-\frac{1}{\sqrt{2010}}.$$

Q6. You have three inscribed squares, with the corners of each inner square at the 1/5'th point along the sides of its outer square. (Thus, for example, $AB = \frac{1}{5}AC$ and $BD = \frac{1}{5}BE$.) The area of the largest square is 81 square feet. What is the area of the smallest square?



Solution: In general, let the area of the big square be a^2, so the side length of the big square is a. Let the corners of each inner square be at the $1/n$'th point along the sides of its outer square. By Pythagorean's theorem, the area of the first smaller square is

$$\|BE\|^2 = \left(\frac{a}{n}\right)^2 + a^2\left(1-\frac{1}{n}\right)^2 = a^2\left(\left(\frac{1}{n}\right)^2 + \left(1-\frac{1}{n}\right)^2\right).$$

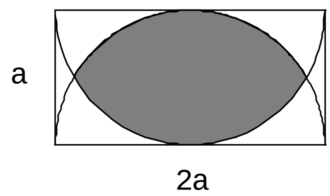
We notice that, at each stage, the smaller square has area of the previous larger square scaled (multiplied) by a factor of $\left(\left(\frac{1}{n}\right)^2 + \left(1-\frac{1}{n}\right)^2\right)$. Therefore the area of the next smallest square is

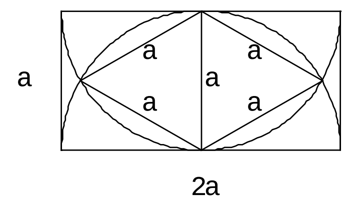
$$a^2\left(\left(\frac{1}{n}\right)^2 + \left(1-\frac{1}{n}\right)^2\right)\left(\left(\frac{1}{n}\right)^2 + \left(1-\frac{1}{n}\right)^2\right) = a^2\left(\left(\frac{1}{n}\right)^2 + \left(1-\frac{1}{n}\right)^2\right)^2.$$

For this particular problem, $a = 9$ and $n = 5$, so the area of the smallest square becomes

$$9^2\left(\left(\frac{1}{5}\right)^2 + \left(1-\frac{1}{5}\right)^2\right)^2 = \frac{81\cdot 17^2}{25^2} = \frac{23409}{625}.$$

Q7. A rectangle has height a and length $2a$. What is the area of the shaded region, which is the intersection of the two semicircles pictured?

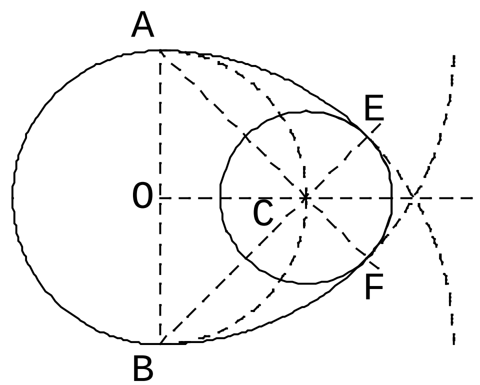




Solution: The area of each equilateral triangle of side length a is $\frac{a}{2}\cdot\frac{\sqrt{3}a}{2}=\frac{\sqrt{3}a^2}{4}$ while the area of each sector of the semicircle (bounded by an arc of a semicircle and two sides of an equilateral triangle) with angle $\pi/3$ is $\frac{\pi a^2}{6}$. The area we want is twice the sum of the areas of two of these sectors minus the area of one equilateral triangle, that is,

$$2\cdot\left[2\cdot\frac{\pi a^2}{6}-\frac{\sqrt{3}a^2}{4}\right]=2a^2\cdot\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right].$$

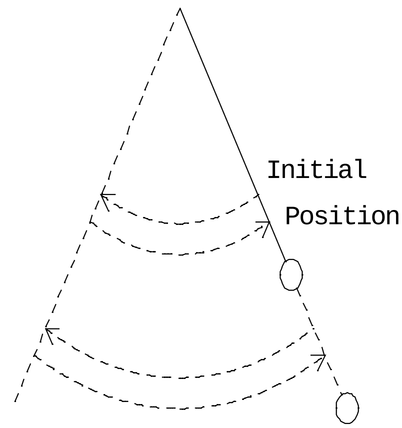
Q8. An egg-shaped figure in the plane is composed of four arcs of circles, designated by $\overset{\frown}{AB}$, $\overset{\frown}{BF}$, $\overset{\frown}{FE}$ and $\overset{\frown}{EA}$, put end to end as indicated in the figure. Knowing that the radius AO has length one, determine the area of the egg figure.



Solution: Notice that the segment AC has length $\sqrt{2}$. The area of the egg is equal to the area of the semicircle ABO (of radius 1) plus the area of the sector BAF (of radius 2) plus the area of the sector ABE (radius 2) plus the area of the quarter circle ECF (radius $2-\sqrt{2}$) minus the area of the triangle ABC (base 2 and height 1). One then obtains the final area being

$$\text{Area of egg} = \frac{\pi}{2} + \frac{\pi \cdot 2^2}{8} + \frac{\pi \cdot 2^2}{8} + \frac{\pi\left(2-\sqrt{2}\right)^2}{4} - \frac{2}{2} = \left(3-\sqrt{2}\right)\pi - 1.$$

Q9. The oscillation period of a pendulum is proportional to the square root of its length (for example, to triple the oscillation period, we multiply the length by nine). Two pendulums of different lengths are released from the initial position shown. The shorter one measures 25 cm, and its oscillation period is one second. The two pendulums are aligned again for the first time after seven seconds in their initial position. Find the length of the longer pendulum assuming its period is a natural number. (Air resistance is neglected.)



Solution: Letting T denote the oscillation period and L the length of a pendulum, we are given that $T = k\sqrt{L}$, where k is the constant of proportionality. To find this constant, the smaller pendulum gives $1 = k\sqrt{25}$, or $k = 1/5$. Since the period of the longer pendulum must be seven seconds, the longer pendulum shows that $7 = \sqrt{L^*}/5$, where L^* denotes the length of the longer pendulum. Therefore the length of the longer pendulum must be $L^* = 35^2 = 1225$ cm.

Q10. Consider the six different numbers obtained by permuting (mixing) the digits of the number 123. The sum of these numbers is 123 + 132 + 213 + 231 + 312 + 321 = 1332. What result would be found if we summed the 720 different numbers obtained by permuting the digits of the number 123456?

Solution: In general, let us permute the digits $1, 2, \ldots, n$, with $n \leq 9$. Notice when one adds up all possible permutations of these digits, there will be $(n-1)!$ of each digit in the ones column, $(n-1)!$ of each digit in the tens column, $(n-1)!$ of each digit in the hundreds column, etc. Now add all columns up, expanding in base 10, to get

$$S = (n-1)!\cdot\left[1+2+\cdots+n\right] + 10^1\cdot(n-1)!\cdot\left[1+2+\cdots+n\right] + \cdots + 10^{n-1}\cdot(n-1)!\cdot\left[1+2+\cdots+n\right]$$

$$= (n-1)!\cdot\frac{(n)(n+1)}{2}\cdot\left[1+10+10^2+\cdots+10^{n-1}\right]$$

$$= \frac{(n+1)!}{2}\cdot\left[\frac{10^n-1}{9}\right].$$

When $n = 6$ this sum becomes

$$\frac{7!}{2}\cdot\frac{999999}{9} = \frac{7!\cdot 111111}{2} = \frac{5040\cdot 111111}{2} = (2520)(111111) = 279999720.$$